# NONSEPARABLE GROWTH OF THE INTEGERS SUPPORTING A MEASURE

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Abstract. Assuming $\mathfrak{b} = \mathfrak{c}$ (or some weaker statement), we construct a compactification $\gamma\omega$ of $\omega$ such that its remainder $\gamma\omega \setminus \omega$ is nonseparable and carries a strictly positive measure.

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## 1. Introduction

We consider here compactifications $\gamma\omega$ of the discrete space $\omega$. Given a compact space $K$, we say that $K$ is a growth of $\omega$ if there is a compactification $\gamma\omega$ with the remainder $\gamma\omega \setminus \omega$ homeomorphic to the space $K$. It is well-known that every separable compactum is a growth of $\omega$.

By a well-known theorem due to Parovičenko [10], under the continuum hypothesis every compact space $K$ of topological weight $\leq \mathfrak{c}$ is an continuous image of the remainder $\beta\omega \setminus \omega$ of the Čech-Stone compactification of $\omega$ and, consequently, $K$ is homeomorphic to the remainder $\gamma\omega \setminus \omega$ of some compactification $\gamma\omega$.

Let $S$ be the Stone space of the measure algebra which may be seen as the quotient of $Bor(2^\omega)$ modulo the ideal of null sets. Then $S$ is a nonseparable compact space that carries a strictly positive (regular probability Borel) measure, i.e. a measure that is positive on every nonempty open subset of $S$. Since the topological weight of $S$ is $\mathfrak{c}$, CH implies that $S$ is a growth of $\omega$. In fact, this may be done in such a way that the canonical measure on $S$ is defined by the asymptotic density defined for subsets of $\omega$, see Frankiewicz and Gutek [5].

Dow and Hart [3] proved that the space $S$ is not a growth of $\omega$ if one assumes the Open Coloring Axiom (OCA). Therefore it seems to be interesting to investigate whether one can always, in the usual set theory, construct a compactification $\gamma\omega$ such that the remainder $\gamma\omega \setminus \omega$ is nonseparable but carries a strictly positive measure.

If a compact space carries a strictly positive measure then it satisfies *ccc*; the converse does not hold which was already demonstrated by Gaifman [6]. Later Bell [2], van Mill [9], Todorčević [11] constructed several interesting examples of compactifications of $\omega$

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having nonseparable *ccc* remainders; cf. Todorčević [12]. It seems that the structure of all those examples exclude the possibility that *ccc* could be strenghten to saying that the remainder in question supports a measure, see e.g. Lemma 3.2 in Džamonja and Plebanek [4].

Before we state our main result we need to fix some notation and terminology concerning the set-theoretic assumption that we use. We denote by $\lambda$ the usual product measure on the Cantor set $2^\omega$. Let $\mathcal{E}$ be an ideal of subsets $A$ of $2^\omega$ for which $\lambda(\overline{A}) = 0$. Recall that the covering number $\mathrm{cov}(\mathcal{E})$ is the least cardinality of a covering of $2^\omega$ by sets from $\mathcal{E}$; cf. Bartoszyński and Shelah [1] for cardinal invariants of the ideal $\mathcal{E}$. We shall sometimes write $\kappa_0 = \mathrm{cov}(\mathcal{E})$ for simplicity. As usual, $[\kappa_0]^{\le\omega}$ denotes the family of all countable subsets of $\kappa_0$. Recall that $\mathrm{cof}[\kappa_0]^{\le\omega}$, the cofinality of this partially ordered set, is the least size of a family $\mathcal{J} \subseteq [\kappa_0]^{\le\omega}$ such that every countable subset of $\kappa_0$ is contained in some $J \in \mathcal{J}$. Our set-theoretic assumption (*) involves also $\mathfrak{b}$, the familiar bounding number and reads as follows.

$$(*) \quad \text{writing } \kappa_0 = \mathrm{cov}(\mathcal{E}) \text{ we have } \mathrm{cof}[\kappa_0]^{\le\omega} \le \mathfrak{b}.$$

**Theorem 1.1.** *Assuming (*) there is a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is not separable but carries a strictly positive regular probability Borel measure.*

Note that $(*)$ holds whenever $\mathfrak{b} = \mathfrak{c}$ or $\kappa_0 = \omega_1$. We do not know whether Theorem 1.1 can be proved in the usual set theory. In connection with the result of Dow and Hart mentioned above it is worth remarking that under OCA, $\mathfrak{b} = \omega_2$ and we can further assume $\omega_2 = \mathfrak{c}$ (see Moore [8]) so the compactification we construct here may exist even when the Stone space of the measure algebra is not a growth of $\omega$.

We remark that in our proof of Theorem 1.1 we construct $\gamma\omega$ such that $\gamma\omega\setminus\omega$ supports a measure $\mu$ of countable Maharam type (meaning that $L_1(\mu)$ is separable). We might, however, modify the construction so that the resulting $\mu$ will be of type $\kappa$.

The paper is organized as follows. In section 2 we formulate Theorem 1.1 in terms of subalgebras of $P(\omega)$ and finitely additive measures defined on them, see Theorem 2.2. Section 3 describes an inductive construction using (*) that leads to 2.2. The key lemma showing that the inductions works is postponed to the final section 4.

## 2. Compactifications and Boolean algebras

We denote by *fin* a family of finite subsets of $\omega$. In the sequel, we shall consider Boolean algebras (of sets) $\mathfrak{A}$ such that $fin \subseteq \mathfrak{A} \subseteq P(\omega)$. Every such an algebra $\mathfrak{A}$ determines a compactification $K_\mathfrak{A}$ of $\omega$, where $K_\mathfrak{A}$ may be seen as the Stone space of all ultrafilters on $\mathfrak{A}$. Then the algebra $\mathrm{Clop}(K^*_\mathfrak{A})$ of the clopen subsets of the remainder

$K^*_{\mathfrak{A}} = K_{\mathfrak{A}} \setminus \omega$ is isomorphic to the quotient algebra $\mathfrak{A}/\mathit{fin}$. Hence $K^*_{\mathfrak{A}}$ is not separable if and only if $\mathfrak{A}/\mathit{fin}$ is not $\sigma$-centred.

Given an algebra $\mathfrak{A}$ such that $\mathit{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$, we shall consider finitely additive probability measures $\mu$ on $\mathfrak{A}$ that vanish on finite sets. Such a measure $\mu$ defines, via the Stone isomorphism, a finitely additive measure $\widehat{\mu}$ on $\mathrm{Clop}(K_{\mathfrak{A}})$. Then $\widehat{\mu}$ extends uniquely to a regular probability Borel measure $\overline{\mu}$ on $K_{\mathfrak{A}}$ such that $\overline{\mu}(K^*_{\mathfrak{A}}) = 1$. Note that the resulting Borel measure will be strictly positive whenever $\mu$ has the property mentioned in the following definition.

**Definition 2.1.** *If $\mathit{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ and $\mu$ is finitely additive measure on $\mathfrak{A}$ then we shall say that $\mu$* is almost strictly positive *if for every $A \in \mathfrak{A}$, $\mu(A) = 0$ if and only if $A \in \mathit{fin}$.*

We can summarise our preliminary remarks and conclude that Theorem 1.1 is an immediate consequence of the following result.

**Theorem 2.2.** *Assume $(*)$. There exists a Boolean algebra $\mathfrak{A}$ such that $\mathit{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ and a finitely additive probability measure $\mu$ on $\mathfrak{A}$ such that*
*(a) $\mathfrak{A}/\mathit{fin}$ is not $\sigma$-centred;*
*(b) $\mu$ is almost strictly positive.*

We shall prove Theorem 2.2 by inductive construction, gradually enlarging Boolean algebras and extending measures. If $X \subseteq \omega$ then $\mathfrak{A}[X]$ stands for an algebra generated by $\mathfrak{A} \cup \{X\}$. It is easy to check that $\mathfrak{A}[X] = \{(A \cap X) \cup (A' \setminus X) \colon A, A' \in \mathfrak{A}\}$.

If $\mu$ is finitely additive on $\mathfrak{A}$ and $Z \subseteq \omega$ then we write

$$\mu_*(Z) = \sup\{\mu(A) \colon A \in \mathfrak{A}, A \subseteq Z\}, \quad \mu^*(Z) = \inf\{\mu(A) \colon A \in \mathfrak{A}, A \supseteq Z\},$$

for the corresponding inner- and outer-measure. Recall the following well-known fact about extension of measures, see Łoś and Marczewski [7].

**Proposition 2.3.** *The formula*

$$\widetilde{\mu}\big((A \cap X) \cup (A' \setminus X)\big) = \mu_*(A \cap X) + \mu^*(A' \setminus X),$$

*defines an extension of $\mu$ to a finitely additive measure $\widetilde{\mu}$ on $\mathfrak{A}[X]$.*

## 3. A construction

In this section we prove Theorem 2.2. We start by fixing a countable dense subset $D$ of the Cantor space $2^\omega$ playing the role of $\omega$ (and *fin* stands for finite subsets of $D$). Let $\mathfrak{A}_0$ be the subalgebra of $P(D)$ generated by $\{C \cap D : C \in \mathrm{Clop}(2^\omega)\}$ and all finite subsets of $D$. Further let $\mu_0$ be a measure on $\mathfrak{A}_0$ defined by

$$\mu_0(C \triangle F) = \lambda(C) \text{ for } C \in \mathrm{Clop}(2^\omega) \text{ and } F \in \mathit{fin}.$$

Clearly, $\mu_0$ is well-defined and almost strictly positive on $\mathfrak{A}_0$. Note that $\mu_0$ is nonatomic, in the sense that every element of $\mathfrak{A}_0$ may be written as a finite union of sets of arbitrary small measure.

Our construction requires a certain bookkeeping of families of sequences in $\mathfrak{A}_0$. It will be convenient to use the following definition.

**Definition 3.1.** *A countable family $\mathcal{S} \subseteq (\mathfrak{A}_0)^\omega$ will be called an $s-$*family *if for every $S = (S(k))_k \in \mathcal{S}$*

*(i) $S(0) \supseteq S(1) \supseteq \dots$ and $\bigcap_k S(k) = \emptyset$;*

*(ii) $\lim_k \lambda_0(S(k)) = 0$.*

Let $\mathcal{S}$ be a countable infinite $s-$family together with some fixed enumeration $\mathcal{S} = \{S_n : n \in \omega\}$. Then for any $\varphi \in \omega^\omega$ we write

$$X_\varphi(\mathcal{S}) = \bigcup_{n\in\omega} S_n\big(\varphi(n)\big).$$

We shall write below $\kappa_0 = \mathrm{cov}(\mathcal{E})$ and $\kappa = \mathrm{cof}[\kappa_0]^{\leq\omega}$ for simplicity.

Choose a family $\{Z_\alpha \colon \alpha < \kappa_0\}$ of closed subsets of $2^\omega$ such that $\lambda(Z_\alpha) = 0$ for every $\alpha < \kappa_0$ and $\bigcup_{\alpha<\kappa_0} Z_\alpha = 2^\omega$. To every $Z_\alpha$ we associate $S_\alpha \in \mathfrak{A}_0^\omega$ as follows. Fix a bijection $d\colon \omega \to D$. Write $Z_\alpha$ as an intersection of a decreasing sequence of $C_{\alpha,k} \in \mathrm{Clop}(2^\omega)$ and set

$$S_\alpha(k) = C_{\alpha,k} \cap D \setminus \{d(0), \dots, d(k-1)\}.$$

Let $\mathcal{J}$ be a cofinal family in $[\kappa_0]^{\leq\omega}$ of cardinality $\kappa$. Given $J \in \mathcal{J}$, write $\mathcal{S}^J = \{S_\alpha : \alpha \in J\}$. Then $\mathcal{S}^J$ is an $s-$family in our terminology. For any $\varphi \in \omega^J$ we put

$$X_\varphi(\mathcal{S}^J) = \bigcup_{\alpha\in J} S_\alpha(\varphi(\alpha)).$$

**Lemma 3.2.** *Assume $(*)$. There are a family $(\mathfrak{A}_\xi)_{\xi<\kappa}$ of Boolean subalgebras of $P(D)$ and a family $(\mu_\xi)_{\xi<\kappa}$, where every $\mu_\xi$ is a finitely additive measure defined on $\mathfrak{A}_\xi$, such that, writing $\mathfrak{A} = \bigcup_{\xi<\kappa} \mathfrak{A}_\xi$, we have*

*(i) $|\mathfrak{A}_\xi| < \kappa$ for every $\xi < \kappa$;*

*(ii) $\mu_\xi$ is almost strictly positive on $\mathfrak{A}_\xi$;*

*(iii) $\mu_\eta | \mathfrak{A}_\xi = \mu_\xi$ whenever $\xi < \eta < \kappa$;*

*(iv) for every $J \in \mathcal{J}$ there is $\varphi \in \omega^J$ such that $X_\varphi(\mathcal{S}^J) \in \mathfrak{A}$ and $\omega \setminus X_\varphi(\mathcal{S}^J)$ is infinite.*

*Proof.* Enumerate $\mathcal{J}$ as $(J_\xi)_{1\leq\xi<\kappa}$. We start from $\mathfrak{A}_0$ and $\mu_0$ defined at the beginning of this section. Fix $\xi < \kappa$. Given $\mathfrak{A}_\beta$ and $\mu_\beta$ for $\beta < \xi$ we apply Lemma 4.3 from section 4 to the algebra $\mathfrak{B} = \cup_{\beta<\xi}\mathfrak{A}_\beta$, the measure $\nu$ on $\mathfrak{B}$ which extends all $\mu_\beta$, $\beta < \xi$ and an

$s-$family $\mathcal{S} = \mathcal{S}^{J_\xi}$. Then Lemma 4.3 gives us a suitable $X = X_\varphi(\mathcal{S}^{J_\xi})$ and we let $\mathfrak{A}_\xi$ be $\mathfrak{B}[X]$ and $\mu_\xi$ be an extension of $\nu$ to an almost strictly positive measure on $\mathfrak{A}_\xi$. $\square$

Now we check that the algebra $\mathfrak{A}$ satisfying (i)-(iv) of Lemma 3.2 is the one we are looking for.

**Lemma 3.3.** *If $\mathfrak{A}$ is an algebra is in 3.2 then $\mathfrak{A}$ carries an almost strictly positive finitely additive probability measure and $\mathfrak{A}/fin$ is not $\sigma$-centred.*

*Proof.* It is clear that if we let $\mu$ be the unique measure on $\mathfrak{A}$ extending all $\mu_\xi$'s then $\mu$ is as required.

Checking that $\mathfrak{A}/fin$ is not $\sigma$-centred amounts to verifying that if $\{p_k : k \in \omega\}$ is a family of nonprincipial ultrafilters on $\mathfrak{A}$ then there is an infinite $A \in \mathfrak{A}$ such that $A \notin p_k$ for every $k$.

Clearly every $p_k$ defines the unique $t_k \in 2^\omega$ which is in the intersection

$$\bigcap\{C \in \mathrm{Clop}(2^\omega) : C \cap D \in p_k\}.$$

Then $t_k \in Z_{\alpha_k}$ for some $\alpha_k < \kappa_0$. An $s-$family $\{S_{\alpha_k} : k \in \omega\}$ is contained in $\mathcal{S}^J$ for some $J \in \mathcal{J}$. Take $X_\varphi(\mathcal{S}^J)$ as in 3.2(iv). Then $A = D \setminus X_\varphi(\mathcal{S}^J)$ is an infinite set lying outside every $p_k$, as required. $\square$

## 4. Key lemma

We shall now prove an auxiliary result, stated below as Lemma 4.3, showing that the inductive construction of Lemma 3.2 can be carried out. We follow here the notation introduced in section 3; in particular, the notion of an $s$-family was introduced in 3.1. Recall also that, given an $s$-family $\mathcal{S} = \{S_i : i \in I\}$, any $J \subseteq I$ and a function $\varphi \in \omega^J$, we write $X_\varphi(\mathcal{S}) = \bigcup_{i\in J} S_i(\varphi(i))$. We sometimes write $X_\varphi$ rather than $X_\varphi(\mathcal{S})$ if $\mathcal{S}$ is clear from the context.

**Lemma 4.1.** *Let $\mathcal{S} = \{S_0, \dots, S_{n-1}\}$ be a fixed finite s-family. For a given infinite set $A \subseteq \omega$ we put*

$$\Phi^A_n = \{\varphi \in \omega^n \colon |A \setminus X_\varphi(\mathcal{S})| = \omega\}.$$

*If we consider $\Phi^A_n$ with the natural partial order then it has finitely many minimal elements.*

*Proof.* It is pretty obvious that for every $\varphi \in \Phi^A_n$ there is minimal $\varphi' \in \Phi^A_n$ such that $\varphi' \leq \varphi$.

Fix an infinite set $A \subseteq \omega$. We prove the lemma by induction on $n$. For $n = 1$ it is trivial since $\Phi^A_1 \subseteq \omega$ is well-ordered. Assume that for any $k < n$ and any infinite $B \subseteq \omega$ the family $\Phi^B_k$ has finitely many minimal elements.

Fix some minimal $\varphi_0 \in \Phi_n^A$. Consider any non-empty set $I \subseteq n$ of size less than $n$ and any function $\psi \in \omega^I$ such that $\psi \leq \varphi_0 \restriction I$ and $A \setminus X_\psi$ is infinite (it is important that there is only finitely many such $\psi$ and sets $I \subseteq n$). By inductive assumption applied to $A \setminus X_\psi$ and an $s-$family indexed by $n \setminus I$, there exist finitely many minimal elements in $\Phi_{n\setminus I}^{A\setminus X_\psi}$ (call them $\varphi^j$). Observe that any minimal element $\varphi \in \Phi_n^A$ is of the form $\varphi = \psi \cup \varphi^j$ for some $j$, so the proof is complete. $\square$

**Lemma 4.2.** *Let $\mathfrak{B} \subseteq P(D)$ be a Boolean algebra of size less than $\mathfrak{b}$ and let $\mathcal{S} = \{S_n : n \in \omega\}$ be a fixed $s-$family in $\mathfrak{B}^\omega$.*

*There exists $g_0 \in \omega^\omega$ such that for any $g \in \omega^\omega$ with $g \geq g_0$, whenever $A \in \mathfrak{B}$ and $A \subseteq X_g$ then $A \subseteq \bigcup_{j \leq N} S_j(g(j))$ for some $N \in \omega$.*

*Proof.* Fix $A \in \mathfrak{B}$. It follows from Lemma 4.1 that for any $n \in \omega$ there exists a finite set $I_n \subseteq \omega$ such that if $\varphi \in \omega^n$ and $|A \setminus X_\varphi| = \omega$ then $(A \setminus X_\varphi) \cap I_n \neq \emptyset$. We inductively define a function $h_A \in \omega^\omega$ such that

$$S_n\big(h_A(n)\big) \cap \bigcup_{j \leq n} I_j = \emptyset,$$

which can be done since $(S_n(k))_k$ is a decreasing sequence with empty intersection.

Now the family of functions $\{h_A : A \in \mathfrak{B}\}$ is of size less than $\mathfrak{b}$ so there exists $g_0 \in \omega^\omega$ such that $h_A \leq^* g_0$ for any $A \in \mathfrak{B}$. We shall check that $g_0$ is as required.

Take any $A \in \mathfrak{B}$ and $g \geq g_0$, and suppose that $A \subseteq X_g$. Let $N \in \omega$ be such that $h_A(n) \leq g(n)$ for any $n \geq N$. Write $\varphi$ for the restriction of $g$ to $N$. Note that the set $A \setminus X_\varphi$ must be finite; indeed, otherwise $\varphi \in \Phi_N^A$ so $J = (A \setminus X_\varphi) \cap I_N \neq \emptyset$. But $I_N \cap S_n(g(n)) = \emptyset$ for every $n \geq N$, hence $J \subseteq A \setminus X_g$ which means that $A$ is not contained in $X_g$, contrary to our assumption.

As $A \setminus X_\varphi$ is finite, the lemma follows. $\square$

We are now ready for the key lemma; recall that the measure $\mu_0$ on $\mathfrak{A}_0$ was introduced in section 3.

**Lemma 4.3.** *Let $\mathfrak{B} \subseteq P(D)$ be a Boolean algebra of size $< \mathfrak{b}$ containing $\mathfrak{A}_0$ and let $\nu$ be a finitely additive almost strictly positive probability measure on $\mathfrak{B}$ extending $\mu_0$. Let $\mathcal{S} = \{S_n : n \in \omega\}$ be a fixed $s-$family contained in $\mathfrak{A}_0^\omega$.*

*There exists $g \in \omega^\omega$ such that for $X_g = X_g(\mathcal{S})$, $\omega \setminus X_g$ is infinite and $\nu$ can be extended to an almost strictly positive measure on $\mathfrak{B}[X_g]$.*

*Proof.* For any $X \subseteq P(\omega)$ we may consider an extension $\widetilde{\nu}$ of $\nu$ to a finitely additive measure on $\mathfrak{B}[X]$ given by the formula as in Proposition 2.3. The plan is to find a function $g \in \omega^\omega$ such that the measure $\widetilde{\nu}$ defined in this way is almost strictly positive

on $\mathfrak{B}[X_g]$. Observe that the extended measure $\widetilde{\nu}$ *will* be almost strictly positive if for every $A \in \mathfrak{B}$

4.3(i) $\nu_*(A \cap X_g) > 0$ whenever $A \cap X_g$ infinite,

4.3(ii) $\nu^*(A \setminus X_\varphi) > 0$ whenever $A \setminus X_\varphi$ infinite,

so we need to find $g$ for which (i) and (ii) are satisfied..

Fix infinite $A \in \mathfrak{B}$ and $n \in \omega$. Consider $\Phi_n^A$ from Lemma 4.1; as this set contains finitely many minimal elements and $\nu$ is almost strictly positive, we can choose $\varepsilon_n^A > 0$ such that $\nu(A \setminus X_\varphi) \geq \varepsilon_n^A$ for every $\varphi \in \Phi_n^A$. Note that the sequence of $\varepsilon_n^A$ is decreasing.

We inductively define $h_A \in \omega^\omega$ so that

$$\nu\Big(S_n\big(h_A(n)\big)\Big) \leq \frac{\varepsilon_n^A}{2^{n+2}}.$$

This can be done since $S_n(k) \in \mathfrak{A}_0$, $\nu|\mathfrak{A}_0 = \mu_0$ and $\lim_{k\to\infty} \mu_0\big(S_n(k)\big) = 0$ by our definition of an $s-$family.

Let $g_0 \in \omega^\omega$ be a function as in Lemma 4.2 Since $|\mathfrak{B}| < \mathfrak{b}$, there exists a function $g_1 \in \omega^\omega$ such that $g_1(n) \geq g_0(n)$ for every $n$ and $h_A \leq^* g_1$ for every $A \in \mathfrak{B}$; say that $h_a(n) \leq g_1(n)$ for every $n \geq N_A$.

Claim. For every $g \geq g_1$, if $A \in \mathfrak{B}$ and $A \setminus X_g$ is infinite then $\nu^*(A \setminus X_g) \geq \frac{1}{2}\varepsilon_{N_A}^A$.

Let $\varphi$ be the restriction of $g$ to $N_A$; write $Y = \bigcup_{n=N_A}^\infty S_n(g(n))$. With this notation we have $A \setminus X_g = (A \setminus X_\varphi) \setminus Y$, where $A \setminus X_\varphi \in \mathfrak{B}$.

Note first that $\nu^*(A \setminus X_g) \geq \nu(A \setminus X_\varphi) - \nu_*(Y)$; indeed if $B \in \mathfrak{B}$, $B \supseteq A \setminus X_g$ then $(A \setminus X_\varphi) \setminus B \subseteq Y$ so

$$\nu(A \setminus X_\varphi) - \nu(B) \leq \nu((A \setminus X_\varphi) \setminus B) \leq \nu_*(Y).$$

Further note that $\nu(A \setminus X_\varphi) \geq \varepsilon_{N_A}^A$ so to verify Claim it is sufficient to check that $\nu_*(Y) \leq \varepsilon_{N_A}^A/2$. Since $g_1 \geq g_0$, it follows from Lemma 4.2 that whenever $B \in \mathfrak{B}$ and $B \subseteq Y$ then $B$ is contained in $\bigcup_{N_A \leq n \leq k} S_n(g(n))$ for some $k$. Therefore, using finite additivity of $\nu$, we get

$$\nu(B) \leq \sum_{n=N_A}^{k} \nu\Big(S_n\big(g(n)\big)\Big) \leq \sum_{n=N_A}^{k} \frac{1}{2^{n+2}}\varepsilon_n^A \leq \frac{1}{2^{N_A+1}}\varepsilon_{N_A}^A \leq \frac{1}{2}\varepsilon_{N_A}^A.$$

This shows that indeed $\nu_*(Y) \leq \varepsilon_{N_A}^A/2$, and the proof of Claim is complete.

Note that by an analogous argument we check that if $g \geq g_1$ then $\nu_*(X_g) \leq 1/2$ so $\omega \setminus X_g$ must be infinite.

We have proved that every $g \geq g_1$ guarantees 4.3(ii) so to complete the proof we need to find $g \geq g_1$ for which 4.3(i) holds.

We again apply a diagonal argument using $|\mathfrak{B}| < \mathfrak{b}$. Given $A \in \mathfrak{B}$, let $f_A(n) = g_1(n)$ if $|A \cap S_n(k)| = \omega$ for every $k$. Otherwise, if $A \cap S_n(k)$ is finite for some $k$ we can take $f_A(n) \geq f_1(n)$ such that $A \cap S_n(f_A(n)) = \emptyset$ (recall that $\bigcap_k S_n(k) = \emptyset$). Finally, let $g \in \omega^\omega$ be a function that eventually dominates every $f_A$, $A \in \mathfrak{B}$. If $A \in \mathfrak{B}$ and $\nu_*(A \cap X_g) = 0$ then $A \cap S_n(g(n))$ is finite for every $n$. Taking $N$ such that $g(n) \geq f_A(n)$ for every $n \geq N$, and writing $\varphi$ for the restriction of $g$ to $N$ we get $A \subseteq X_\varphi$ so $A$ is finite itself. Now $g$ is so that both 4.3(i)-(ii) are satisfied, and the proof is complete. $\square$

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